MODULE - II

CONTINUOUS PROBABILITY DISTRIBUTIONS

CONTINUOUS RANDOM VARIABLE & PROBABILITY DISTRIBUTIONS

PROBABILITY DENSITY FUNCTION

If X is a continuous random variable and $P(a<X<b) = \int_a^b f(x)\,dx$ then $f(x)$ is called probability density function. (pdf) of a continuous random variable provided it should satisfy the condition.

i) $f(x) \geq 0 \;\forall x$ ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

1. Check whether the following function $f(x)$ is a p.d.f

$$f(x) = \begin{cases} 2e^{-2x} & x>0 \\ 0 & x \leq 0. \end{cases}$$

$f(x) = 2e^{-2x} \geq 0$ —① $e^x \geq 0 \;\forall x$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} 2e^{-2x}\,dx.$$

$$= 2\left[\frac{e^{-2x}}{-2}\right]_0^{\infty} = -\left[e^{-\infty} - e^{0}\right] = -(0-1) = 1$$

$e^{-\infty} = 0$

$e^{0} = 1$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$ —② From ① & ② we have the given function is a pdf

U.Q

2. If the probability density function of a random variable is given by $f(x) = \begin{cases} kx^2 & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$

i) find k.

ii) $P(\frac{1}{4}<X<\frac{1}{2})$

iii) $P(X>\frac{2}{3})$

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx.$

$$\Rightarrow \int_0^1 kx^2\,dx = 1 \Rightarrow k\left(\frac{x^3}{3}\right)_0^1 = 1 \Rightarrow \frac{k}{3} = 1$$

$$\Rightarrow \underline{k=3}$$

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ii) $P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)\,dx = \int_{1/4}^{1/2} 3x^2\,dx = 3\left(\frac{x^3}{3}\right)_{1/4}^{1/2}$

$= \left(\frac{1}{2}\right)^3 - \left(\frac{1}{4}\right)^3 = \frac{7}{64}$

iii) $P\left(X > \frac{2}{3}\right) = \int_{2/3}^{\infty} f(x)\,dx = \int_{2/3}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = \int_{2/3}^{1} 3x^2\,dx$

$= 3\left(\frac{x^3}{3}\right)_{2/3}^{1} = \frac{19}{27}$

3. Find k so that the following can serve as pdf; $f(x) = \begin{cases} kxe^{-4x^2} & x>0 \\ 0 & \text{elsewhere} \end{cases}$

$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$

$\Rightarrow \int_0^\infty kxe^{-4x^2}\,dx = 1 \Rightarrow \int_0^\infty k e^{-4u} \frac{1}{2}du$

$x^2 = u.$

$2x\,dx = du.$

$x\,dx = \frac{1}{2}du.$

$\Rightarrow \frac{k}{2}\left[\frac{e^{-4u}}{-4}\right]_0^\infty = 1 \Rightarrow \frac{k}{-8}\left[e^{-\infty} - e^0\right] = 1$

$x=0 \Rightarrow u=0$

$x=\infty \Rightarrow u=\infty$

$\Rightarrow \frac{k}{-8}(0-1) = 1 \Rightarrow \frac{k}{8} = 1 \Rightarrow k = \underline{\underline{8}}.$

CUMULATIVE DISTRIBUTION FUNCTION

If X is ~~the~~ a continuous random variable then the cumulative distribution function denoted by $F(x)$

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx$$

Properties.

- $F(-\infty) = 0$
- $F(\infty) = 1$
- $f(x) = \frac{d}{dx}F(x)$
- $P(a \le X \le b) = F(b) - F(a)$

4.Q

1. If the distribution function of a random variable $F(x) = \begin{cases} 1-\frac{1}{x^2}, & x>1 \\ 0 & x \le 1 \end{cases}$ Find the probabilities that this random variable will take on a value . i) less than 3 ii) between 4 & 5.

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i) $P(X \le 3) = F(3)$

$$F(3) = 1 - \frac{1}{3^2} = 1 - \frac{1}{9} = \frac{8}{9}$$

ii) $P(4 \le X \le 5) = F(b) - F(a)$

$$= F(5) - F(4) = \left(1 - \frac{1}{5^2}\right) - \left(1 - \frac{1}{4^2}\right)$$

$$= \frac{9}{400}$$

2. If a continuous random variable X has a pdf $f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \le 0 \end{cases}$
Find the distribution function of X & Use it to determine the probability that the random variable will take the value less than or equal to 1

$$F(x) = \int_{-\infty}^{x} f(x)\,dx.$$

$x > 0$

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx.$$

$$= \int_{0}^{x} 2e^{-2x}\,dx. = 2\left[\frac{e^{-2x}}{-2}\right]_0^x = -\left[e^{-2x} - e^{0}\right]$$

$$F(x) = 1 - e^{-2x}.$$

$$P(X \le 1) = F(1) = 1 - e^{-2}.$$

3. Find the distribution function of the pdf

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 2 - x & 1 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

Case I : $x < 0$

$$F(x) = 0$$

Case II : $0 < x < 1$

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$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx$$

$$= \int_{0}^{x} f(x)dx = \int_{0}^{x} x\,dx = \left(\frac{x^2}{2}\right)_0^x$$

$$= \frac{1}{2}(x^2 - 0) = \frac{x^2}{2}$$

Case-III : $1 < x < 2$.

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx$$

$$= \int_{0}^{1} f(x)dx + \int_{1}^{x} f(x)dx = \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left(2x - \frac{x^2}{2}\right)_1^x = \frac{1}{2} + 2x - \frac{x^2}{2} - 2 + \frac{1}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

Case IV : $x \geq 2$

$$F(x) = 1$$

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2/2 & 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1 & 1 < x < 2 \\ 1 & x > 2 \end{cases}$$

3. If X is a continuous random variable with pdf given by

$$f(x) = \begin{cases} x/8 & 0 < x < 2 \\ 1/4 & 2 < x < 4 \\ \frac{1}{8}(6-x) & 4 < x < 6 \\ 0 & \text{otherwise.} \end{cases}$$

Case I : $x < 0$

$$F(x) = 0$$

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Case II : $0 < x < 2$

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{x} f(x)dx = \int_{0}^{x} f(x)dx = \int_{0}^{x} \frac{x}{8}dx = \frac{x^2}{16}$$

Case III : $2 < x < 4$

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{x} f(x)dx = \frac{x-1}{4}$$

Case IV : $4 < x < 6$

$$F(x) = \int_{-\infty}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{4} f(x)dx + \int_{4}^{x} f(x)dx = \frac{12x - x^2 - 20}{16}$$

Case V : $x > 6$

$$F(x) = 1$$

MEAN AND VARIANCE

If X is a continuous random variable then its mean or mathematical expectation

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x\, f(x)dx.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx.$$

$$\text{Variance, } \sigma^2 = E[X^2] - E[X]^2$$

1. Let X be a random variable with pdf $f(x) = \begin{cases} \frac{x^2}{3} & -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

find 1) Mean 2) Variance 3) std. deviation 4) $E[4X+5]$

$$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x\, f(x)dx.$$

$$= \int_{-\infty}^{-1} x\, f(x)dx + \int_{-1}^{2} x\, f(x)dx + \int_{2}^{\infty} x\, f(x)dx$$

$$= \int_{-1}^{2} x\, f(x)dx = \int_{-1}^{2} x \cdot \frac{x^2}{3} = \frac{1}{3}\left(\frac{x^4}{4}\right)_{-1}^{2}$$

$$= \frac{1}{12}\left[2^4 - (-1)^4\right] = \frac{1}{12}(16-1) = \frac{15}{12} = \frac{5}{4}$$

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$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-1}^{2} x^2 \cdot f(x)dx = \int_{-1}^{2} x^2 \cdot \frac{x^2}{3} dx = \frac{1}{3}\int_{-1}^{2} x^4 dx.$$

$$= \frac{1}{3}\left(\frac{x^5}{5}\right)_{-1}^{2} = \frac{1}{15}\left(2^5 - (-1)^5\right) = \frac{1}{15}\left[32+1\right] = \frac{33}{15}$$

Variance $= E[X^2] - E[X]^2 = \frac{33}{15} - \left(\frac{5}{4}\right)^2 = \frac{33}{15} - \frac{25}{16} = \frac{51}{80}$

2. Find the value of k and hence find the mean and variance of the distribution $f(x) = kx^2 e^{-x}$ $0 < x < \infty$.

$\int_0^{\infty} e^{-x} x^{n-1} dx = (n-1)!$
$= \Gamma n$

k

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^{\infty} k x^2 e^{-x} dx = k\int_0^{\infty} e^{-x} \cdot x^{3-1} dx.$$

$$\Rightarrow k \cdot 2! = 1 \Rightarrow k = \frac{1}{2}.$$

Mean $= E[X] = \int_{-\infty}^{\infty} x f(x) dx. = \int_0^{\infty} x \cdot k x^2 e^{-x} dx.$

$$= \frac{1}{2}\int_0^{\infty} e^{-x} x^3 dx = \frac{1}{2}\int_0^{\infty} e^{-x} x^{4-1} dx = \frac{(4-1)!}{2} = \frac{3!}{2}$$

$$= \underline{\underline{3}}$$

Variance $= E[X^2] - E[X]^2$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2 \cdot k x^2 e^{-x} dx = k\int_0^{\infty} e^{-x} \cdot x^4 dx$$

$$= k \cdot 4! = \frac{4!}{2} = 12$$

Variance $= 12 - 3^2 = \underline{\underline{3}}$

3. A continuous random variable X has pdf $f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{elsewhere.} \end{cases}$

Find the values of a & b such that

1) $P(X \le a) = P(X > a)$ 2) $P(X > b) = 0.05$

$P(X \le a) = P(X > a)$

$1 - P(X > a) = P(X > a) \Rightarrow 1 = 2P(X > a)$

$$P(X > a) = \frac{1}{2} \Rightarrow \int_a^{\infty} f(x)dx = \frac{1}{2} \Rightarrow \int_a^{1} f(x)dx + \int_1^{\infty} f(x)dx.$$

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$$\int_a^1 3x^2\,dx = \frac{1}{2} \Rightarrow 3\left[\frac{x^3}{3}\right]_a^1 = \frac{1}{2} \Rightarrow \left[x^3\right]_a^1 = \frac{1}{2}$$

$$\Rightarrow 1 - a^3 = \frac{1}{2} \Rightarrow a^3 = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow a = 0.7937$$

2) $P(X > b) = 0.05$

$$\int_b^{\infty} f(x)dx = 0.05 \Rightarrow \int_b^1 3x^2 dx = 0.05 \Rightarrow (x^3)_b^1 = 0.05$$

$$\Rightarrow b^3 = 0.95 \Rightarrow b = 0.983$$

4. A random variable X has the following probability density function $f(x) = kx(2-x)$ $0 < x < 2$ find 1) value of k 2) mean 3) Variance 4) distribution function.

1) $$\int_{-\infty}^{\infty} kx(2-x)dx = 1 \Rightarrow k\int_{-\infty}^{\infty} 2x - x^2\,dx = 1$$

$$\Rightarrow k\int_0^2 (2x - x^2)\,dx = 1 \Rightarrow k\left[2\cdot\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow k\left[4 - \frac{8}{3}\right] = 1 \Rightarrow k\cdot\frac{4}{3} = 1 \Rightarrow k = \frac{3}{4}$$

2) $$\text{Mean} = E[X] = \int_{-\infty}^{\infty} x f(x)dx = \frac{3}{4}\int_0^2 x\cdot x(2-x)dx.$$

$$= \frac{3}{4}\int_0^2 (2x^2 - x^3)dx.$$

$$= \frac{3}{4}\left[2\cdot\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3} - 4\right]$$

$$= \frac{3}{4}\cdot\frac{4}{3} = \underline{\underline{1}}$$

3) $$E[X^2] = \frac{3}{4}\int_0^2 x^3(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx.$$

$$= \frac{3}{4}\left[2\cdot\frac{x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left[8 - \frac{32}{5}\right]$$

$$= \frac{3}{4}\left[\frac{8}{5}\right] = \frac{6}{5}$$

$$\text{Variance} = E[X^2] - E[X]^2 = \frac{6}{5} - 1 = \underline{\underline{\frac{1}{5}}}$$

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UNIFORM DISTRIBUTION [RECTANGULAR DISTRIBUTION]

A continuous random variable X follows uniform distribution over the interval $[\alpha, \beta]$. If its pdf is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

Here α & β are the parameters of uniform distribution.

MEAN

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\,dx = \int_{\alpha}^{\beta} x \cdot \frac{1}{\beta-\alpha}\,dx = \frac{1}{\beta-\alpha}\left[\frac{x^2}{2}\right]_{\alpha}^{\beta}$$

$$= \frac{1}{\beta-\alpha}\left[\frac{\beta^2-\alpha^2}{2}\right] = \frac{(\beta-\alpha)(\beta+\alpha)}{2(\beta-\alpha)} = \frac{\alpha+\beta}{2}.$$

VARIANCE

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{\alpha}^{\beta} x^2 \cdot \frac{1}{\beta-\alpha}\,dx = \frac{1}{\beta-\alpha}\left(\frac{x^3}{3}\right)_{\alpha}^{\beta}$$

$$= \frac{1}{3(\beta-\alpha)}(\beta^3-\alpha^3) = \frac{(\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2)}{3(\beta-\alpha)}$$

$$= \frac{\beta^2+\alpha\beta+\alpha^2}{3}$$

$$E[X]^2 = \left(\frac{\alpha+\beta}{2}\right)^2 = \frac{\alpha^2+2\alpha\beta+\beta^2}{4}$$

$$\text{Variance} = E[X^2] - E[X]^2 = \frac{\alpha^2+\alpha\beta+\beta^2}{3} - \frac{\alpha^2+2\alpha\beta+\beta^2}{4}$$

$$= \frac{\beta^2-2\alpha\beta+\alpha^2}{12} = \frac{(\beta-\alpha)^2}{12} = \frac{(\alpha-\beta)^2}{12}$$

1. If X is uniformly distributed over $(-\alpha, \alpha)$, $\alpha > 0$. Find α so that i) $P(X>1) = \frac{1}{3}$ ii) $P(|X|<1) = P(|X|>1)$

$X \sim U(-\alpha, \alpha)$

$$f(x) = \begin{cases} \frac{1}{2\alpha} & -\alpha < x < \alpha. \\ 0 & \text{otherwise.} \end{cases}$$

i) $P(X>1) = \frac{1}{3} \Rightarrow \int_1^{\alpha} f(x)\,dx = \frac{1}{3} \Rightarrow \int_1^{\alpha} \frac{1}{2\alpha}\,dx = \frac{1}{2\alpha}[x]_1^{\alpha}$

$$\Rightarrow \frac{\alpha-1}{2\alpha} = \frac{1}{3} \Rightarrow 3\alpha-3 = 2\alpha \Rightarrow \alpha = 3$$

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ii) $P(|x|<1) = P(|x|>1)$

$P(|x|<1) = 1 - P(|x|\le 1)$

$2P(|x|<1) = 1 \Rightarrow P(|x|<1) = \frac{1}{2}$

$\Rightarrow P[-1<X<+1] = \frac{1}{2}$

$\Rightarrow \int_{-1}^{1} f(x)\,dx = \frac{1}{2} \Rightarrow \int_{-1}^{1} \frac{1}{2\alpha}\,dx = \frac{1}{2}$

$\Rightarrow \frac{1}{2\alpha}(x)_{-1}^{1} = \frac{1}{2} \Rightarrow \frac{1}{2\alpha}(1+1) = \frac{1}{2}$

$\Rightarrow \frac{1}{\alpha} = \frac{1}{2} \Rightarrow \underline{\underline{\alpha = 2}}$

2. Buses arrived at a specified stop at 15 minute intervals starting at 7 AM. A passenger arrives at the stop at random time between 7 AM and 7.30 AM. Find the probability that he waits i) less than 5 minutes ii) atleast 12 minutes.

$X \sim U(0, 15)$ $\qquad X \rightarrow$ Waiting time.

$$f(x) = \begin{cases} \frac{1}{15} & , 0<x<15 \\ 0 & \text{otherwise} \end{cases}$$

i) $P(X<5) = \int_{0}^{5} f(x)\,dx = \int_{0}^{5} \frac{1}{15}\,dx = \frac{1}{15}(x)_{0}^{5} = \underline{\underline{\frac{1}{3}}}$

ii) $P(X>12) = \int_{12}^{15} f(x)\,dx = \int_{12}^{15} \frac{1}{15}\,dx = \frac{1}{15}(x)_{12}^{15} = \frac{3}{15}$

$= \underline{\underline{\frac{1}{5}}}$

3. If X is a uniformly distributed random variable with mean 1 and variance $\frac{4}{3}$. Find $P(|X-2|<2)$

Mean $= 1 \Rightarrow \frac{\alpha+\beta}{2} = 1 \Rightarrow \alpha+\beta = 2$

Variance $= \frac{4}{3} \Rightarrow \frac{(\alpha-\beta)^2}{12} = \frac{4}{3} \Rightarrow (\alpha-\beta)^2 = 16 \Rightarrow \alpha-\beta = \pm 4$

$\begin{cases} \alpha+\beta = 2 \\ \alpha-\beta = 4 \end{cases} \Rightarrow \alpha = 3, \quad \beta = -1$

$\begin{cases} \alpha+\beta = 2 \\ \alpha-\beta = -4 \end{cases} \Rightarrow \alpha = -1 \quad \beta = 3$

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since $\alpha < x < \beta$ we have $-1 < x < 3$

$$f(x) = \begin{cases} \frac{1}{4} & -1 < x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

$$P(|X-2| < 2) \Rightarrow P[-2 < X-2 < 2]$$

$$\Rightarrow P[-2+2 < X-2+2 < 2+2]$$

$$\Rightarrow P[0 < X < 4]$$

$$= \int_0^3 f(x)\,dx = \int_0^3 \frac{1}{4}\,dx$$

$$= \frac{1}{4}(x)_0^3 = \frac{3}{4}.$$

4. The thickness of photo resist applied to wafers in semiconductor manufacturing at a particular location on the wafer is uniformly distributed between 0.205 & 0.215 micrometers. What thickness is exceeded by 10% of the wafers.

$X \sim U(0.205, 0.215)$

$$f(x) = \begin{cases} \frac{1}{0.01} & 0.205 < x < 0.215 \\ 0 & \text{otherwise.} \end{cases}$$

$$P(X > a) = \frac{10}{100} = 0.1$$

$$\int_a^{0.215} f(x)\,dx = 0.1 \Rightarrow \int_a^{0.215} \frac{1}{0.01}\,dx = \frac{1}{0.01}\left[x\right]_a^{0.215}$$

$$\Rightarrow \frac{1}{0.01}\left[0.215 - a\right] = 0.1 \Rightarrow 0.215 - a = 0.001$$

$$\Rightarrow a = 0.214$$

5. A wire of length a is divided into 2 parts. The part of random length X is uniformly distributed. Then find mean, variance & $E[X(a-X)]$

$X \sim U(0,a)$

$$f(x) = \begin{cases} \frac{1}{a} & 0 < x < a \\ 0 & \text{otherwise.} \end{cases}$$

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$$\text{Mean} = E[X] = \int_0^a x\cdot f(x)\,dx = \int_0^a x\cdot\frac{1}{a}\,dx = \frac{1}{a}\cdot\left(\frac{x^2}{2}\right)_0^a = \frac{a}{2}$$

$$\text{Variance} = E[X^2] - E[X]^2$$

$$E[X^2] = \int_0^a x^2 f(x)\,dx = \frac{1}{a}\cdot\left(\frac{x^3}{3}\right)_0^a = \frac{a^2}{3}$$

$$\text{Variance} = \frac{a^2}{3} - \left(\frac{a}{2}\right)^2 = \underline{\underline{\frac{a^2}{12}}}$$

$$E[X(a-X)] = E[aX - X^2] = a\,E[X] - E[X^2]$$

$$= a\cdot\frac{a}{2} - \frac{a^2}{3} = \underline{\underline{\frac{a^2}{6}}}$$

EXPONENTIAL DISTRIBUTION

A continuous random variable X is said to follow exponential distribution if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & x>0,\ \beta>0 \\ 0 & \text{otherwise.} \end{cases}$$

β is the parameter

MEAN

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} x f(x)\,dx = \int_0^{\infty} x\cdot\frac{1}{\beta} e^{-\frac{x}{\beta}}\,dx$$

$$= \frac{1}{\beta}\int_0^{\infty} x e^{-\frac{x}{\beta}}\,dx = \frac{1}{\beta}\left[x\left(\frac{e^{-x/\beta}}{-1/\beta}\right) - (1)\left(\frac{e^{-x/\beta}}{(-1/\beta)^2}\right)\right]_0^{\infty}$$

$$= \frac{1}{\beta}\left[(0-0) - \left(0 - \frac{1}{1/\beta^2}\right)\right] = \frac{\beta^2}{\beta} = \underline{\underline{\beta}}$$

VARIANCE

$$E[X^2] = \int_0^{\infty} x^2\cdot f(x)\,dx = \int_0^{\infty} x^2\cdot\frac{1}{\beta} e^{-x/\beta}\,dx$$

$$= \frac{1}{\beta}\int_0^{\infty} x^2 e^{-x/\beta}\,dx = \frac{1}{\beta}\left\{(x^2)\left(\frac{e^{-x/\beta}}{-1/\beta}\right) - (2x)\left(\frac{e^{-x/\beta}}{(-1/\beta)^2}\right) + (2)\left(\frac{e^{-x/\beta}}{(-1/\beta)^3}\right)\right\}_0^{\infty}$$

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$$= \frac{1}{\beta}\left\{(0-0+0)-(0-0-2\beta^3)\right\}$$

$$= 2\beta^2$$

Variance $= E[X^2] - E[X]^2 = 2\beta^2 - \beta^2 = \beta^2$

Note

$$f(x) = \begin{cases} \lambda e^{-x\lambda} & x>0, \\ 0 & \text{otherwise} \end{cases}$$ Here λ is the parameter.

1. The time in hours required to repair a machine is exponentially distributed with mean 20. What is the probability that the required time i) Exceeds 30 hrs ii) Between 16 hrs and 24 hrs.

Mean $= \beta = 20$

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

i) $P(\text{Exceeds 30 hrs}) = P(X>30) = \int_{30}^{\infty} f(x)\,dx = \frac{1}{20}\int_{30}^{\infty} e^{-x/20}\,dx.$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{30}^{\infty} = -\left[e^{-\infty} - e^{-3/2}\right] = e^{-10}$$

ii) $P(\text{Between 16 hrs \& 24 hrs})$

$$= P(16 < X\ 24) = \int_{16}^{24} f(x)\,dx.$$

$$= \int_{16}^{24} \frac{1}{20} e^{-x/20}\,dx = \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_{16}^{24}$$

$$= -\left[e^{-6/5} - e^{-4/5}\right]$$

2. The amount of time that a surveillance camera will run without having to be reset is a random variable having exponential distribution with mean 50 days. Find the probability that such a camera will i) have to be reset in less than 20 days ii) not have to be reset in atleast 60 days.

$\beta = 50$ $$f(x) = \begin{cases} \frac{1}{50} e^{-x/50} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

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i) P(less than 20 days) $= P(X<20) = \int_0^{20} f(x)\,dx$.

$$= \int_0^{20} \frac{1}{50} e^{-x/50}\,dx = \frac{1}{50}\left[\frac{e^{-x/50}}{\frac{-1}{50}}\right]_0^{20}$$

$$= -\left[e^{-2/5} - e^{0}\right] = 1 - e^{-2/5}$$

ii) P(atleast 60 days) $= P(X \geq 60)$

$$= \int_{60}^{\infty} \frac{1}{50} e^{-x/50}\,dx$$

$$= \frac{1}{50}\left[\frac{e^{-x/50}}{-1/50}\right]_{60}^{\infty} = -\left[e^{-\infty} - e^{-6/5}\right]$$

$$= e^{-6/5}$$

3. Let the mileage (in thousands of miles) of a particular tyre by a random variable X having p.d.f

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20} & x>0 \\ 0 & \text{otherwise.} \end{cases}$$

Find the probability that one of these will last i) anywhere from 16,000 to 24000 ii) Atleast 30,000 miles iii) mean & variance.

X follows exponential distribution with $\beta = 20$

i) $P(16 < X < 24) = \int_{16}^{24} f(x)\,dx = 0.148$

ii) $P(X > 30) = \int_{30}^{\infty} f(x)\,dx = 0.2231$

iii) Mean $= \beta = 20$

Variance $= \beta^2 = 400$

4. Life time for a battery is exponentially distributed. 40% of such batteries do not last longer than 1000 hours. Mr kumar purchased such a battery which is already used for 500 hrs. What is the probability that it will last another 1000 hours.?

60% of batteries last longer than 1000 hrs

$$P(X>1000) = 0.6 \implies \int_{1000}^{\infty} f(x)\,dx = 0.6$$

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$$\Rightarrow \frac{1}{\beta}\int_{1000}^{\infty} e^{-x/\beta}\,dx = 0.6 \Rightarrow -\left[e^{-x/\beta}\right]_{1000}^{\infty} = 0.6 \Rightarrow -\left[e^{-\infty} - e^{-1000/\beta}\right] = 0.6$$

$$\Rightarrow e^{-1000/\beta} = 0.6 \Rightarrow \frac{-1000}{\beta} = \log(0.6) \Rightarrow \beta = 4507.57$$

$$P(500 < X < 1500) = \int_{500}^{1500} f(x)\,dx = \underline{0.17807}$$

RESULT

Exponential distribution satisfies memory-less property

$$P[X > s+t \mid X > s] = P[X > t]$$

$$P[X > s+t \mid X > s] = \frac{P[(X > s+t) \cap X > s]}{P[X > s]}$$

$$= \frac{P[X > s+t]}{P[X > s]} \quad \text{—— (1)}$$

$$P[X > s+t] = \int_{s+t}^{\infty} f(x)\,dx.$$

$$= \int_{s+t}^{\infty} \frac{1}{\beta} e^{-x/\beta}\,dx = \frac{1}{\beta}\left[\frac{e^{-x/\beta}}{-1/\beta}\right]_{s+t}^{\infty}$$

$$= \frac{1}{\beta}\left[0 - \frac{e^{-(s+t)\beta}}{-1/\beta}\right] = e^{-(s+t)/\beta} \quad \text{—— (2)}$$

$$P[X > s] = \frac{1}{\beta}\int_{s}^{\infty} e^{-x/\beta}\,dx. = \frac{1}{\beta}\left[\frac{e^{-x/\beta}}{-1/\beta}\right]_{s}^{\infty} = e^{-s/\beta} \quad \text{—— (3)}$$

$$P[X > s+t \mid X > s] = \frac{e^{-(s+t)/\beta}}{e^{-s/\beta}} = e^{-t/\beta} = L.H.S$$

$$R.H.S = P[X > t] = \int_{t}^{\infty} \frac{1}{\beta} e^{-x/\beta}\,dx = \frac{1}{\beta}\left[\frac{e^{-x/\beta}}{-1/\beta}\right]_{t}^{\infty}$$

$$= e^{-t/\beta}$$

$$L.H.S = R.H.S$$

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NORMAL DISTRIBUTION

A continuous random variable X is said to follow normal distribution with mean μ and variance σ^2 if its pdf is of the form

$$f(x,\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad -\infty < x < \infty$$

Properties

1. The normal curve is bell shaped
2. The normal curve is symmetrical about the line $x=\mu$
3. X-axis is an asymptote to the normal curve.
4. The mean, median and mode of normal distribution coincide.
5. The total area under the normal curve is unity. Therefore area of the normal curve from $x=-\infty$ to μ is $\frac{1}{2}$ and μ to ∞ is $\frac{1}{2}$.

STANDARD NORMAL DISTRIBUTION

The normal distribution with mean $\mu=0$ and standard deviation $\sigma=1$ is called standard normal distribution. Its pdf is

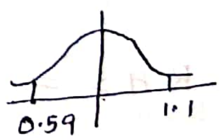
$$F(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad -\infty < z < \infty$$

$$Z \sim N(0,1) \qquad Z = \frac{x-\mu}{\sigma}$$

1. Let X be a continuous random variable with mean $\mu = 4.35$ $\sigma = 0.59$ Find 1) $P(4<X<5)$ 2) $P(X>5.5)$

$\mu = 4.35 \quad \sigma = 0.59 \quad Z = \frac{x-\mu}{\sigma}$.

1) $P(4<X<5) = P\left(\frac{4-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{5-\mu}{\sigma}\right)$



$= P\left(\frac{4-4.35}{0.59} < Z < \frac{5-4.35}{0.59}\right)$

$= P(-0.59 < Z < 1.10)$

$= P(-0.59 < Z < 0) + P(0 < Z < 1.10)$

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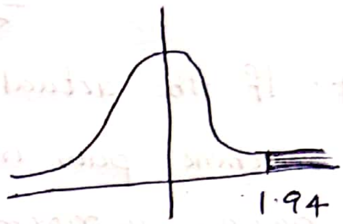
$= 0.2224 + 0.3643 = 0.5867$

2. $P(X > 5.5) = P\left(\frac{X-\mu}{\sigma} > \frac{5.5-\mu}{\sigma}\right) = P(Z > 1.94)$

$= 0.5 - P(0 < Z < 1.94)$

$= 0.5 - 0.4738$

$= \underline{\underline{0.0262}}$

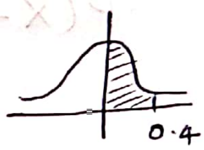


2. The weekly wages of 1000 workman are normally distributed around a mean of Rs 70 & std deviation of Rs 5. Estimate the number of workers whose weekly wages

1) between 70 & 72 2) More than 75 3) less than 63

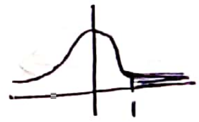
$\mu = 70 \quad \sigma = 5$

1) $P(70 < X < 72) = P(0 < Z < 0.4) = 0.1554$



No: of workers $= 1000 \times 0.1554 = 155.4 \approx 155$.

2) $P(X > 75) = P(Z > 1)$



$= 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413 = 0.1587$

No: of workers $= 1000 \times 0.1587 = 158.7 \approx 159$

3. $P(X < 63) = P(Z < -1.4)$

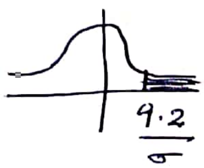
$= 0.5 - P(-1.4 < Z < 0)$

$= 0.5 - P(0 < Z < 1.4)$

$= 0.5 - 0.4192 = 0.0808$

No: of workers $= 1000 \times 0.0808 = 80.8 \approx 81$

3. The time for super glue to set can be treated as a random variable having normal distribution with $\mu = 30$sec. Find std deviation if probability is 0.2 that it will take on a value greater than 39.2 sec.



$\mu = 30$ Sec.

$P(X > 39.2) = 0.2 \Rightarrow P\left(Z > \frac{9.2}{\sigma}\right) = 0.2$

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$$0.5 - P\left(0<z<\frac{9.2}{\sigma}\right) = 0.2$$

$$P\left(0<z<\frac{9.2}{\sigma}\right) = 0.3$$

$$\frac{9.2}{\sigma} = 0.845 \implies \sigma = 10.887$$

4. If the actual amount of instant coffee which a filling machine puts into "6-ounce" jars is a random variable having a normal distribution with standard deviation $\sigma = 0.05$ ounce and if only 3% of jars are to contain less than 6 ounce of coffee, what must be the mean fill of these jars?

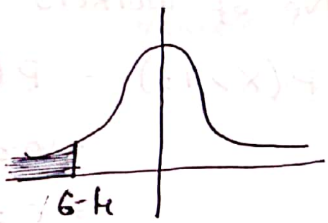
μ = Mean $\qquad \sigma = 0.05$

$$P(X<6) = 0.03 \implies P\left(\frac{X-\mu}{\sigma} < \frac{6-\mu}{0.05}\right) = 0.03$$

$$\implies P\left(z < \frac{6-\mu}{0.05}\right) = 0.03$$

$$\implies 0.5 - P\left(-\left(\frac{6-\mu}{.05}\right) < z < 0\right) = 0.03$$

$$\implies P\left(-\left(\frac{6-\mu}{0.05}\right) < z < 0\right) = 0.47$$



$$\implies -\left(\frac{6-\mu}{0.05}\right) = 1.88$$

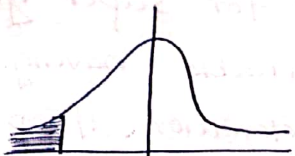
$$\implies \mu = 6.094$$

5. In a normal distribution 17% of the items are below 30 & 17% of the items are below 60. find mean & standard deviation.

$P[X<30] = 0.17 \qquad P[X>60] = 0.17$

$$P\left[\frac{X-\mu}{\sigma} < \frac{30-\mu}{\sigma}\right] = 0.17$$

$$P\left[z < \frac{30-\mu}{\sigma}\right] = 0.17$$



$$0.5 - P\left[0<z<\frac{30-\mu}{\sigma}\right] = 0.17$$

$$P\left[-\left(\frac{30-\mu}{\sigma}\right)<z<0\right] = 0.33 \implies -\left(\frac{30-\mu}{\sigma}\right) = 0.955$$

$$\implies \mu - 30 = 0.955\sigma \quad \text{—(1)}$$

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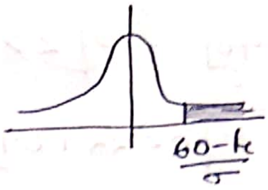
$P[X > 60] = 0.17$



$P\left[z > \frac{60-\mu}{\sigma}\right] = 0.17$

$0.5 - P\left[0 < z < \frac{60-\mu}{\sigma}\right] = 0.17$

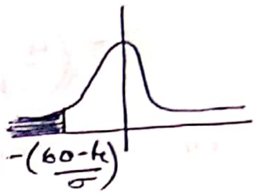
$P\left[0 < z < \frac{60-\mu}{\sigma}\right] = 0.33 \Rightarrow \frac{60-\mu}{\sigma} = 0.955$

$\Rightarrow -\mu + 60 = 0.955\sigma$ — (2)

From (1) & (2) $\mu = 45$ & $\sigma = 15.7$

6. In a large group of man 5% are under 60 inches in height and 40% are between 60 and 65. Assuming normal distribution find the mean and standard deviation.

$P[X < 60] = 0.05 \Rightarrow P\left[z < \frac{60-\mu}{\sigma}\right] = 0.05$



$0.5 - P\left[-\left(\frac{60-\mu}{\sigma}\right) < z < 0\right] = 0.05$

$P\left[-\left(\frac{60-\mu}{\sigma}\right) < z < 0\right] = 0.45$

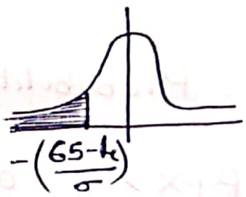
$-\left(\frac{60-\mu}{\sigma}\right) = 1.64 \Rightarrow 60-\mu = -1.64\sigma$ — (1)

$P[60 < X < 65] = 0.4$ $\quad P[X < 65] = P[X<60] + P[60<X<65]$

$= 0.05 + 0.4$

$P[X < 65] = 0.45 \Rightarrow P\left[z < \frac{65-\mu}{\sigma}\right] = 0.45$

$0.5 - P\left[-\left(\frac{65-\mu}{\sigma}\right) < z < 0\right] = 0.45$



$P\left[-\left(\frac{65-\mu}{\sigma}\right) < z < 0\right] = 0.05$

$-\left(\frac{65-\mu}{\sigma}\right) = 0.13 \Rightarrow 65-\mu = -0.13\sigma$ — (2)

From (1) & (2)

$$60-\mu = -1.64\sigma$$
$$65-\mu = -0.13\sigma$$

(1) − (2) $\quad -5 = -1.51\sigma \Rightarrow \sigma = \underline{3.31}$

$\mu = \underline{\underline{65.42}}$

7. The marks obtained in mathematics by 1000 students are normally distributed with mean $\mu = 78$ and $\sigma = 11$. Determine. i) How many students got marks above 90% ii) What is the highest mark obtained by the lowest 10% of students.

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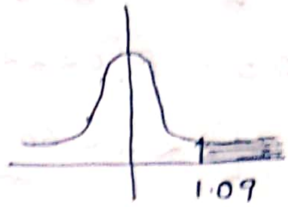
i) $P[X>90] = P\left[z > \frac{90-k}{\sigma}\right] = P[z>1.09]$

$0.5 - P[0<z<1.09] = 0.5 - 0.3621$

$= 0.1379$

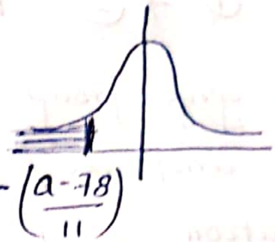
No: of students $= 1000 \times 0.1379 = 138$



ii) $P[X<a] = 0.1 \Rightarrow P\left[z < \frac{a-k}{\sigma}\right] = 0.1$

$P\left[z < \frac{a-78}{11}\right] = 0.1$

$0.5 - P\left[-\left(\frac{a-78}{11}\right) < z < 0\right] = 0.1$



$P\left[-\left(\frac{a-78}{11}\right) < z < 0\right] = 0.4$

$-\left(\frac{a-78}{11}\right) = 1.28 \Rightarrow a = 63.92$

8. In a competitive examination 5000 students have appeared for a paper in engineering mathematics. Their average marks were 62 and standard deviation 12. If there are only 100 vacancies find the minimum marks that one should score in order to get selected

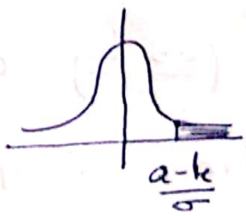
$k = 62 \quad \sigma = 12.$

$P[X>a]$ Let a be the minimum marks that one should score in order to get selected

Probability to be selected $= \frac{1}{5000} \times 100 = 0.02$

$P[X>a] = 0.02 \Rightarrow P\left[z > \frac{a-k}{\sigma}\right] = 0.02$

$P\left[z > \frac{a-62}{12}\right] = 0.02$



$0.5 - P\left[0<z<\frac{a-62}{12}\right] = 0.02$

$P\left[0<z<\frac{a-62}{12}\right] = 0.48$

$\Rightarrow \frac{a-62}{12} = 2.06 \Rightarrow a = 6.72$

$\underline{\underline{a = 6.72}}$

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CONTINUOUS BIVARIATE DISTRIBUTIONS

If X and Y are two random variables (continuous) then $P[X=x, Y=y] = P(x,y) = f(x,y)$ is called the joint probability distribution of (X,Y) if it satisfies the following conditions.

1) $f(x,y) \geq 0 \quad \forall (x,y)$

2. $\iint_R f(x,y)\,dx\,dy = 1$

$\rightarrow P[a \leq X \leq b, c \leq Y \leq d] = \int_a^b \int_c^d f(x,y)\,dy\,dx.$

MARGINAL DISTRIBUTION

If X & Y are continuous random variables then the marginal pdf of X and Y are given by

Marginal pdf of $X = P_X(x) = f_X(x) = \int_y f(x,y)\,dy$

$P_Y(y) = f_Y(y) = \int_x f(x,y)\,dx.$

EXPECTED VALUES

$E[X] = \int_x x f_X(x)\,dx.$

$E[XY] = \int_x \int_y f(x,y)\,dy\,dx.$

$E[Y] = \int_y y f_Y(y)\,dy$

1. The joint pdf of X & Y is given by $f(x,y) = \begin{cases} kxy, & 0<x<4, \ 1<y<5 \\ 0 & \text{otherwise} \end{cases}$

i) Find k ii) Find the marginal pdf of X & Y

iii) Evaluate $P(X \geq 3, Y \leq 2)$

i) We have $\int_0^4 \int_1^5 kxy\,dy\,dx = 1 \Rightarrow \int_0^4 kx\left(\frac{y^2}{2}\right)_1^5 dx = 1$

$\Rightarrow \int_0^4 \frac{kx}{2}(25-1)\,dx = 1 \Rightarrow 12k\int_0^4 x\,dx = 1$

$\Rightarrow k = \frac{1}{96}$

ii) Marginal pdf of X

$f_X(x) = \int_y f(x,y)\,dy = \int_1^5 \frac{xy}{96}\,dy = \frac{x}{8}$

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Marginal pdf of y

$$f_Y(y) = \int_x f(x,y)\,dx = \int_0^4 \frac{xy}{96}\,dx$$

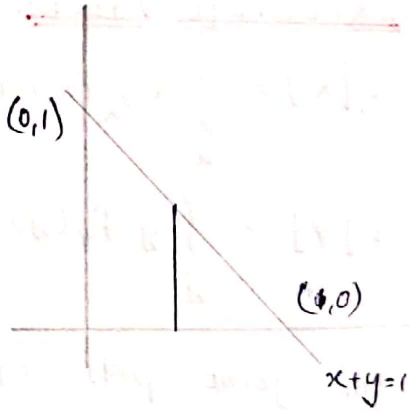
$$= \frac{y}{96}\int_0^4 x\,dx = \frac{y}{96}\cdot\left(\frac{x^2}{2}\right)_0^4 = \underline{\underline{\frac{y}{12}}}$$

iii $$P(X \geq 3,\ y \leq 2) = \int_3^4\int_1^2 \frac{xy}{96}\,dy\,dx$$

$$= \frac{1}{96}\int_3^4 x\cdot\left(\frac{y^2}{2}\right)_1^2 dx = \frac{1}{96}\int_3^4 \frac{3}{2}x\,dx$$

$$= \underline{\underline{\frac{7}{128}}}$$

2. If $f(x,y) = k(1-x-y)$ $x>0,\ y>0$ $x+y<1$ is the joint pdf of $f(x,y)$. Find i) k ii) the marginal distribution iii) Examine whether $X + Y$ are independent



$$\int_{R_x}\int_{R_y} f(x,y)\,dy\,dx = 1$$

$$\int_0^1\int_0^{1-x} k(1-x-y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^1 k\left(y - xy - \frac{y^2}{2}\right)_0^{1-x} dx = 1$$

$$\Rightarrow \int_0^1 k\left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right] dx = 1$$

$$\Rightarrow \int_0^1 k\left[1 - x - x + x^2 - \frac{1}{2} + \frac{2x}{2} - \frac{x^2}{2}\right] dx = 1$$

$$\Rightarrow \int_0^1 k\left[\frac{1}{2} - x + \frac{x^2}{2}\right] dx = 1 \Rightarrow k\left[\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6}\right]_0^1 = 1$$

$$\Rightarrow k\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = 1 \Rightarrow \frac{k}{6} = 1 \Rightarrow k = \underline{\underline{6}}$$

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ii) Marginal distribution of x.

$$f_X(x) = \int_{R_y} f(x,y)\,dy = \int_0^{1-x} 6(1-x-y)\,dy$$

$$= 6\left[y - xy - \frac{y^2}{2}\right]_0^{1-x} = 6\left[(1-x) - x(1-x) - \frac{(1-x)^2}{2}\right]$$

$$= 6\left[\frac{1}{2} - x + \frac{x^2}{2}\right] = 3\left[x^2 - 2x + 1\right]$$

$$= 3(x-1)^2.$$

$$f_X(x) = 3(x-1)^2 \qquad 0 < x < 1$$

Marginal distribution of y

$$f_Y(y) = \int_x f(x,y)\,dx. = \int_0^{1-y} 6(1-x-y)\,dx.$$

$$f_Y(y) = 3(y-1)^2 \qquad 0 < y < 1$$

iii) If X and Y are independent the $f_X(x) \cdot f_Y(y) = f(x,y)$

$3(x-1)^2 \cdot 3(y-1)^2 \neq f(x,y)$, so X and Y are not independent

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